Aequat. Math. 87 (2014), 105–123

0001-9054/14/010105-19
published online February 8, 2013
DOI 10.1007/s00010-013-0185-y

Aequationes Mathematicae

Some generalization of the quadratic and Wilson's functional equation

Radosław Łukasik

Abstract. We find the solutions $f, g, h\colon G \to X$, $\varphi\colon G \to \mathbb{K}$ of each of the functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|\varphi(y)g(x) + |K|h(y), \quad x, y \in G,$$

where $(G, +)$ is an abelian group, K is a finite, abelian subgroup of the automorphism group of G, X is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Mathematics Subject Classification (2000). 39B82.

Keywords. Quadratic functional equation, Wilson's functional equation.

1. Introduction

The generalization of the quadratic functional equation

$$f(x + y) + f(x + \sigma y) = 2f(x) + 2f(y), \quad x, y \in G,$$

where σ is automorphism of an abelian group G such that $\sigma^2 = id_G$, $f, g\colon G \to \mathbb{C}$, was investigated by Stetkær [13].

In another work [14] he solved the functional equation

$$\frac{1}{N}\sum_{n=0}^{N-1} f(z + \omega^n \zeta) = g(z) + h(\zeta), \quad z, \zeta \in \mathbb{C},$$

where $N \in \{2, 3, \ldots\}$, ω is the primitive Nth root of unity, $f, g, h\colon \mathbb{C} \to \mathbb{C}$ are continuous.

Łukasik [8] showed the solution of the functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|g(x) + |K|h(y), \quad x, y \in S,$$

where $(S, +)$ is an abelian semigroup, K is a finite subgroup of the automorphism group of S, $(H, +)$ is an abelian group.

Birkhäuser

The functional equation

$$\sum_{\lambda\in K} f(x+\lambda y)=|K|g(x)h(y),\quad x,y\in G,$$

where $(G,+)$ is an abelian group, K is a finite subgroup of the automorphism group on G, $f,g,h\colon G\to\mathbb{C}$, was studied by Förg-Rob and Schwaiger [5], Gajda [6], Stetkær [11,12], Badora [2].

2. Main result

Throughout the present paper, we assume that X is a linear space over the field $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$, $(G,+)$ is an abelian group, K is a finite, abelian subgroup of the automorphism group of G, $L:=\operatorname{card} K$.

We give the complete solution of the following functional equation

$$\sum_{\lambda\in K} f(x+\lambda y)=L\varphi(y)g(x)+Lh(y),\quad x,y\in S.$$

In this work we use the following two theorems and one lemma:

Theorem 1 (Shinya [9, Corollary 3.12], Kannappan [7], Czerwik [4], Sinopoulos [10], Chojnacki [3]). *Let $(G,+)$ be an abelian, locally compact, Hausdorff topological group, K be a compact Hausdorff topological transformation group of G acting by automorphisms on G. Let further $d\lambda$ be a normalized Haar measure on K. If $\varphi\in C(G)$ is a nonzero solution of*

$$\int_{\lambda\in K}\varphi(x+\lambda y)d\lambda=\varphi(x)\varphi(y),\quad x,y\in G,$$

then there exists a continuous homomorphism $\chi\colon G\to\mathbb{C}^*$ such that

$$\varphi(x)=\int_{\lambda\in K}\chi(\lambda x)d\lambda,\quad x\in G.$$

If φ is bounded, then χ may be taken as a unitary character.

Lemma 1 [1, Lemma 14.1]. *Let Ω be a nonempty set, $n\in\mathbb{N}$. Functions $f_1,\dots,f_n\colon\Omega\to\mathbb{C}$ are linearly dependent if and only if for all $x_1,\dots,x_n\in\Omega$*

$$\begin{vmatrix} f_1(x_1) & f_1(x_2) & \dots & f_1(x_n)\\ f_2(x_1) & f_2(x_2) & \dots & f_2(x_n)\\ \vdots & \vdots & \ddots & \vdots\\ f_n(x_1) & f_n(x_2) & \dots & f_n(x_n)\end{vmatrix}=0.$$

Theorem 2 [8, Theorem 5]. *Let $(S,+)$ be an abelian semigroup, K be a finite subgroup of the automorphism group of S, $L :=$ card K, $(H,+)$ be an abelian group uniquely divisible by $L!$. Then the function $f\colon S \to H$ satisfies the equation*

$$\sum_{\lambda\in K} f(x+\lambda y) = Lf(x), \quad x,y \in S$$

if and only if there exist k-additive, symmetric mappings $A_k\colon S^k \to H$, $k \in \{1,\dots L-1\}$ and $A_0 \in H$ such that

$$f(x) = A_0 + A_1(x) + \cdots + A_{L-1}(x,\cdots,x), \quad x \in S,$$
$$\sum_{\lambda\in K} A_k(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i}) = 0, \quad x,y \in S,\ 1 \le i \le k \le L-1.$$

First we show some lemmas.

Lemma 2. *Let $m\colon G \to \mathbb{C}^*$ be a homomorphism, $K_1 \subset K$ be a set such that $\{m\circ\lambda :\ \lambda \in K_1\}$ is linearly independent. If for some $\mu \in K$*

$$m(\mu x) = \sum_{\lambda\in K_1} a_\lambda m(\lambda x), \quad x \in G, \tag{1}$$

then $m\circ\mu = m\circ\lambda$ for some $\lambda \in K_1$.

Proof. Let

$$m(\mu x) = \sum_{\lambda\in K_1} a_\lambda m(\lambda x), \quad x \in G.$$

Therefore, for $x, y \in G$, we have

$$\sum_{\lambda\in K_1} a_\lambda m(\lambda y)m(\lambda x) = m(\mu(y+x)) = m(\mu y)m(\mu x) = m(\mu y)\sum_{\lambda\in K_1} a_\lambda m(\lambda x),$$

which means that

$$0 = \sum_{\lambda\in K_1} a_\lambda[m(\lambda y) - m(\mu y)]m(\lambda x).$$

From the linear independence we obtain

$$0 = a_\lambda[m(\lambda y) - m(\mu y)], \quad y \in G, \quad \lambda \in K_1,$$

since there exists $\lambda \in K_1$ such that $a_\lambda \neq 0$,

$$m(\lambda y) = m(\mu y), \quad y \in G.$$

□

Lemma 3. *Let $m\colon G \to \mathbb{C}^*$ be a homomorphism. Then the set $K_0 := \{\lambda \in K : m\circ\lambda = m\}$ is a subgroup of K.*

Lemma 4. *Let* $K_0 := \{\lambda \in K:\ m\circ\lambda = m\}$, $K_1 \subset K$ *be a minimal set such that* $K = K_0 \circ K_1$. *Then* K_1 *is a maximal set such that the set* $\{m\circ\lambda:\ \lambda\in K_1\}$ *is linearly independent.*

Proof. In view of Lemma 2, if $\{m\circ\lambda:\ \lambda\in K_1\}$ is linearly dependent, then $m\circ\lambda = m\circ\mu$ for some $\lambda,\mu\in K_1$, so we get $\lambda\circ\mu^{-1}\in K_0$, which means that $\lambda\in K_0\circ\mu$, which contradicts the definition of K_1.

On the other hand each $\lambda\in K$ has the form $\lambda=\lambda_0\circ\lambda_1$, where $\lambda_0\in K_0$, $\lambda_1\in K_1$, hence $m\circ\lambda = m\circ\lambda_0\lambda_1 = m\circ\lambda_1$, which gives us the maximality of K_1. □

Theorem 3. *Let* $\varphi\colon G\to\mathbb{C}$, $\varphi\neq 0$, *satisfy the equation*

$$\sum_{\lambda\in K}\varphi(x+\lambda y) = L\varphi(x)\varphi(y),\quad x,y\in G. \tag{2}$$

Then there exist a homomorphism $m\colon G\to\mathbb{C}^*$, $\beta_\lambda\in\mathbb{C}^*, b_\lambda\in G, \lambda\in K_1$, *such that*

$$\varphi(x) = \frac{1}{L}\sum_{\lambda\in K} m(\lambda x),\quad x\in G, \tag{3}$$

$$\sum_{\lambda\in K_1}\beta_\lambda m(\mu b_\lambda) = \begin{cases} |K_1|, & \mu\in K_0\\ 0, & \mu\notin K_0\end{cases} \tag{4}$$

$$m(x) = \sum_{\lambda\in K_1}\beta_\lambda\varphi(x+b_\lambda),\quad x\in G, \tag{5}$$

where $K_0 := \{\lambda\in K:\ m\circ\lambda = m\}$, $K_1\subset K$ *is a minimal set such that* $K = K_0\circ K_1$.

Proof. If we take a discrete topology on group $(G,+)$ and a counting measure on K divided by L, then the assumptions of Theorem 1 are fulfilled. Hence there exists a homomorphism $m\colon G\to\mathbb{C}^*$ satisfying (3).

Let K_0 and K_1 be as in the statement of this theorem. From the linear independence of the set $\{m\circ\lambda:\ \lambda\in K_1\}$ and Lemma 1 there exist $b_\lambda\in G, \lambda\in K_1$ such that the matrix $[m(\lambda b_\mu)]_{\lambda,\mu\in K_1}$ has a nonzero determinant. Hence, there exist $\beta_\lambda\in\mathbb{C}^*, \lambda\in K_1$, which satisfy (4).
We notice that

$$\begin{aligned} m(x) &= \frac{1}{|K_0|}\sum_{\mu\in K_0} m(\mu x) = \frac{1}{|K_0|\cdot|K_1|}\sum_{\mu\in K} m(\mu x)\sum_{\lambda\in K_1}\beta_\lambda m(\mu b_\lambda)\\ &= \sum_{\lambda\in K_1}\beta_\lambda\frac{1}{L}\sum_{\mu\in K} m(\mu(x+b_\lambda)) = \sum_{\lambda\in K_1}\beta_\lambda\varphi(x+b_\lambda),\quad x\in G,\end{aligned}$$

which proves (5). □

Now we prove a generalization of Wilson's functional equation.

Theorem 4. *Assume that X is complex. Functions $f: G \to X, f \neq 0$, $\varphi: G \to \mathbb{C}$ satisfy the equation*

$$\sum_{\lambda \in K} f(x + \lambda y) = L\varphi(y) f(x), \quad x, y \in G, \tag{6}$$

if and only if there exist a homomorphism $m: G \to \mathbb{C}^*$, $A_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda: G^k \to X, k \in \{1, \ldots |K_0| - 1\}, \lambda \in K_1$ such that

$$\varphi(x) = \frac{1}{L} \sum_{\lambda \in K} m(\lambda x), \quad x \in G, \tag{7}$$

$$f(x) = \sum_{\lambda \in K_1} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \ldots, x) \right], \quad x \in G, \tag{8}$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \ldots, x, \underbrace{\mu y, \ldots, \mu y}_{i}) = 0, \quad x, y \in G, \ \lambda \in K_1, \ 1 \leq i \leq k \leq |K_0| - 1, \tag{9}$$

where $K_0 := \{\lambda \in K : \ m \circ \lambda = m\}$, $K_1 \subset K$ is a minimal set such that $K = K_0 \circ K_1$.

Proof. It is easy to check that if functions f and φ satisfy conditions (7), (8), (9), then they satisfy Eq. (6).

Assume that f and φ satisfy Eq. (6). Since $f \neq 0$, $\varphi \neq 0$. Note that for $x, y, z \in G$ we have

$$\begin{aligned} L \sum_{\mu \in K} \varphi(y + \mu z) f(x) &= \sum_{\mu \in K} \sum_{\lambda \in K} f(x + \lambda(y + \mu z)) = \sum_{\mu \in K} \sum_{\lambda \in K} f(x + \lambda y + \mu z) \\ &= \sum_{\lambda \in K} L\varphi(z) f(x + \lambda y) = L^2 \varphi(z)\varphi(y) f(x). \end{aligned}$$

Taking $x \in G$ such that $f(x) \neq 0$ we obtain that φ satisfies Eq. (2). In view of Theorem 3 we get (7). Now we show that

$$|K_1| \sum_{\sigma \in K_0} f(x + \sigma y) = \sum_{\rho \in K} m(\rho y) \sum_{\nu \in K_1} \beta_\nu f(x + \rho^{-1} b_\nu), \quad x, y \in G. \tag{10}$$

Indeed, we have the following sequence of identities

$$\begin{aligned} |K_1| \sum_{\sigma \in K_0} f(x + \sigma y) &= \sum_{\lambda \in K} \sum_{\mu \in K_1} \beta_\mu m(\lambda b_\mu) f(x + \lambda y) \\ &= \sum_{\lambda \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \beta_\mu \beta_\nu \varphi(\lambda b_\mu + b_\nu) f(x + \lambda y) \\ &= \sum_{\lambda \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \sum_{\rho \in K} \frac{1}{L} \beta_\mu \beta_\nu f(x + \lambda y + \rho\lambda b_\mu + \rho b_\nu) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\rho\in K}\sum_{\mu\in K_1}\sum_{\nu\in K_1}\beta_\mu\beta_\nu\varphi(y+\rho b_\mu)f(x+\rho b_\nu)\\
&= \sum_{\rho\in K}\sum_{\nu\in K_1}\beta_\nu m(\rho^{-1}y)f(x+\rho b_\nu)\\
&= \sum_{\rho\in K} m(\rho y)\sum_{\nu\in K_1}\beta_\nu f(x+\rho^{-1}b_\nu),\quad x,y\in G.
\end{aligned}
$$

For each $\tau\in K_1$ we define $g_\tau\colon G\to X$ by the formula

$$
g_\tau(x) := \frac{1}{Lm(\tau x)}\sum_{\nu\in K_1}\beta_\nu\sum_{\sigma\in K_0} f(x+\sigma\tau^{-1}b_\nu),\quad x\in G.
$$

From equality (10) we obtain

$$
\begin{aligned}
&Lm(\tau x+\tau y)\sum_{\sigma\in K_0} g_\tau(x+\sigma y)\\
&\quad= \sum_{\sigma\in K_0} Lm(\tau(x+\sigma y))g_\tau(x+\sigma y)\\
&\quad= \sum_{\sigma\in K_0}\sum_{\nu\in K_1}\beta_\nu\sum_{\rho\in K_0} f(x+\sigma y+\rho\tau^{-1}b_\nu)\\
&\quad= \sum_{\nu\in K_1}\beta_\nu\sum_{\rho\in K_0}\sum_{\sigma\in K_0} f(x+\sigma y+\sigma\rho\tau^{-1}b_\nu)\\
&\quad= \sum_{\nu\in K_1}\beta_\nu\sum_{\rho\in K_0}\frac{1}{|K_1|}\sum_{\mu\in K} m(\mu(y+\rho\tau^{-1}b_\nu))\sum_{\sigma\in K_1}\beta_\sigma f(x+\mu^{-1}b_\sigma)\\
&\quad= \sum_{\mu\in K} m(\mu y)\sum_{\nu\in K_1}\beta_\nu\sum_{\rho\in K_0}\frac{1}{|K_1|}m(\mu\rho\tau^{-1}b_\nu)\sum_{\sigma\in K_1}\beta_\sigma f(x+\mu^{-1}b_\sigma)\\
&\quad= \sum_{\mu\in K} m(\mu y)\frac{|K_0|}{|K_1|}\sum_{\nu\in K_1}\beta_\nu m(\mu\tau^{-1}b_\nu)\sum_{\sigma\in K_1}\beta_\sigma f(x+\mu^{-1}b_\sigma)\\
&\quad= |K_0|\sum_{\mu\in K_0} m(\mu\tau y)\sum_{\sigma\in K_1}\beta_\sigma f(x+\tau^{-1}\mu^{-1}b_\sigma)\\
&\quad= |K_0|m(\tau y)\sum_{\sigma\in K_1}\beta_\sigma\sum_{\mu\in K_0} f(x+\mu\tau^{-1}b_\sigma)\\
&\quad= |K_0|\cdot Lm(\tau y)m(\tau x)g_\tau(x),\quad x\in G.
\end{aligned}
$$

Hence

$$
\sum_{\sigma\in K_0} g_\tau(x+\sigma y) = |K_0|g_\tau(x),\quad x\in G,\quad \tau\in K_1. \tag{11}
$$

In view of Theorem 2, for each $\lambda \in K_1$ there exist $A_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda \colon S^k \to X, k \in \{1, \dots |K_0| - 1\}$ such that

$$g_\lambda(x) = A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x), \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_{i}) = 0, \quad x, y \in G,\ 1 \leq i \leq k \leq |K_0| - 1.$$

We observe that

$$\begin{aligned} L \sum_{\lambda \in K_1} m(\lambda x) g_\lambda(x) &= \sum_{\lambda \in K_1} \sum_{\nu \in K_1} \beta_\nu \sum_{\sigma \in K_0} f(x + \sigma \lambda^{-1} b_\nu) \\ &= \sum_{\nu \in K_1} \beta_\nu \sum_{\lambda \in K} f(x + \lambda b_\nu) = \sum_{\nu \in K_1} \beta_\nu L \varphi(b_\nu) f(x) \\ &= L m(0) f(x) = L f(x), \quad x \in G, \end{aligned}$$

which ends the proof. □

Theorem 5. *Assume that X is real. Functions $f \colon G \to X, f \neq 0, \varphi \colon G \to \mathbb{R}$ satisfy the equation*

$$\sum_{\lambda \in K} f(x + \lambda y) = L \varphi(y) f(x), \quad x, y \in G, \tag{12}$$

if and only if there exist a homomorphism $m \colon G \to \mathbb{C}^*, A_0^\lambda \in X$, $B_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda, B_k^\lambda \colon G^k \to X, k \in \{1, \dots |K_0| - 1\}, \lambda \in K_1$ such that

$$\varphi(x) = \frac{1}{L} \sum_{\lambda \in K} m(\lambda x), \quad x \in G, \tag{13}$$

$$\begin{aligned} f(x) = \sum_{\lambda \in K_1} \Bigg(&Re\ m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] \\ &- Im\ m(\lambda x) \left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \dots, x) \right] \Bigg), \quad x \in G, \end{aligned} \tag{14}$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \leq i \leq k \leq |K_0| - 1, \tag{15}$$

$$\sum_{\mu \in K_0} B_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \leq i \leq k \leq |K_0| - 1, \tag{16}$$

where $K_0 := \{\lambda \in K : \ m \circ \lambda = m\}$, $K_1 \subset K$ is a minimal set such that $K = K_0 \circ K_1$.

Proof. It is easy to check that if functions f and φ satisfy conditions (13), (14), (15), then they satisfy Eq. (12).

Assume that f and φ satisfy Eq. (12). Since $f \neq 0$, $\varphi \neq 0$. We observe that for $x, y, z \in G$ we have

$$L \sum_{\mu \in K} \varphi(y+\mu z) f(x) = \sum_{\mu \in K} \sum_{\lambda \in K} f(x+\lambda(y+\mu z)) = \sum_{\mu \in K} \sum_{\lambda \in K} f(x+\lambda y+\mu z)$$
$$= \sum_{\lambda \in K} L\varphi(z) f(x+\lambda y) = L^2 \varphi(z)\varphi(y) f(x).$$

Taking $x \in G$ such that $f(x) \neq 0$ we obtain that φ satisfies Eq. (2). In view of Theorem 3 we get (13). From equalities (4) and (5) we have

$$|K_1| \sum_{\sigma \in K_0} f(x+\sigma y) = \sum_{\rho \in K_1} \sum_{\nu \in K_1} \mathrm{Re}\,(\beta_\nu m(\rho y)) \sum_{\sigma \in K_0} f(x+\sigma\rho^{-1} b_\nu), \quad x, y \in G, \tag{17}$$

$$0 = \sum_{\rho \in K_1} \sum_{\nu \in K_1} \mathrm{Im}\,(\beta_\nu m(\rho y)) \sum_{\sigma \in K_0} f(x+\sigma\rho^{-1} b_\nu), \quad x, y \in G. \tag{18}$$

Indeed, for $x, y \in G$ we have

$$\begin{aligned}|K_1| \sum_{\sigma \in K_0} f(x+\sigma y) &= \sum_{\lambda \in K} \mathrm{Re}\left(\sum_{\mu \in K_1} \beta_\mu m(\lambda b_\mu)) f(x+\lambda y\right)\\
&= \sum_{\lambda \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \mathrm{Re}\,(\beta_\mu \beta_\nu) \varphi(\lambda b_\mu + b_\nu) f(x+\lambda y)\\
&= \sum_{\lambda \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \sum_{\rho \in K} \frac{1}{L} \mathrm{Re}\,(\beta_\mu \beta_\nu) f(x+\lambda y+\rho\lambda b_\mu+\rho b_\nu)\\
&= \sum_{\rho \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \mathrm{Re}\,(\beta_\mu \beta_\nu) \varphi(y+\rho b_\mu) f(x+\rho b_\nu)\\
&= \sum_{\rho \in K} \sum_{\nu \in K_1} \mathrm{Re}\,(\beta_\nu m(\rho^{-1} y)) f(x+\rho b_\nu)\\
&= \sum_{\rho \in K_1} \sum_{\nu \in K_1} \mathrm{Re}\,(\beta_\nu m(\rho y)) \sum_{\sigma \in K_0} f(x+\sigma\rho^{-1} b_\nu),\end{aligned}$$

and

$$\begin{aligned}0 &= \sum_{\lambda \in K} \mathrm{Im}\left(\sum_{\mu \in K_1} \beta_\mu m(\lambda b_\mu)\right) f(x+\lambda y)\\
&= \sum_{\lambda \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \mathrm{Im}\,(\beta_\mu \beta_\nu) \varphi(\lambda b_\mu + b_\nu) f(x+\lambda y)\\
&= \sum_{\lambda \in K} \sum_{\mu \in K_1} \sum_{\nu \in K_1} \sum_{\rho \in K} \frac{1}{L} \mathrm{Im}\,(\beta_\mu \beta_\nu) f(x+\lambda y+\rho\lambda b_\mu+\rho b_\nu)\end{aligned}$$

$$= \sum_{\rho\in K}\sum_{\mu\in K_1}\sum_{\nu\in K_1} \mathrm{Im}\,(\beta_\mu\beta_\nu)\varphi(y+\rho b_\mu)f(x+\rho b_\nu)$$
$$= \sum_{\rho\in K}\sum_{\nu\in K_1} \mathrm{Im}\,(\beta_\nu m(\rho^{-1}y))f(x+\rho b_\nu)$$
$$= \sum_{\rho\in K_1}\sum_{\nu\in K_1} \mathrm{Im}\,(\beta_\nu m(\rho y))\sum_{\sigma\in K_0} f(x+\sigma\rho^{-1}b_\nu),$$

For each $\tau \in K_1$ we define $g_\tau, h_\tau\colon G \to X$ by the formula

$$g_\tau(x) := \sum_{\nu\in K_1} \mathrm{Re}\,\frac{\beta_\nu}{Lm(\tau x)}\sum_{\sigma\in K_0} f(x+\sigma\tau^{-1}b_\nu), \quad x\in G,$$
$$h_\tau(x) := \sum_{\nu\in K_1} \mathrm{Im}\,\frac{\beta_\nu}{Lm(\tau x)}\sum_{\sigma\in K_0} f(x+\sigma\tau^{-1}b_\nu), \quad x\in G.$$

From equalities (17) and (18) we obtain

$$\sum_{\sigma\in K_0} g_\tau(x+\sigma y) = |K_0|g_\tau(x), \quad x,y\in G, \quad \tau\in K_1, \tag{19}$$
$$\sum_{\sigma\in K_0} h_\tau(x+\sigma y) = |K_0|h_\tau(x), \quad x,y\in G, \quad \tau\in K_1. \tag{20}$$

Indeed, we have the following sequence of identities

$$|K_1|\sum_{\lambda\in K_0} g_\tau(x+\lambda y)$$
$$= |K_1|\sum_{\lambda\in K_0}\sum_{\nu\in K_1} \mathrm{Re}\,\frac{\beta_\nu}{Lm(\tau(x+\lambda y))}\sum_{\sigma\in K_0} f(x+\lambda y+\sigma\tau^{-1}b_\nu)$$
$$= \sum_{\nu\in K_1} \mathrm{Re}\,\frac{\beta_\nu}{Lm(\tau(x+y))}\sum_{\sigma\in K_0}|K_1|\sum_{\lambda\in K_0} f(x+\lambda(y+\sigma\tau^{-1}b_\nu))$$
$$= \sum_{\nu\in K_1} \mathrm{Re}\,\frac{\beta_\nu}{Lm(\tau(x+y))}\sum_{\sigma\in K_0}\sum_{\rho\in K_1}\sum_{\mu\in K_1} \mathrm{Re}\,(\beta_\mu m(\rho(y+\sigma\tau^{-1}b_\nu)))$$
$$\cdot\sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu)$$
$$= \sum_{\nu\in K_1} \mathrm{Re}\,\frac{|K_0|\beta_\nu}{Lm(\tau(x+y))}\sum_{\rho\in K_1}\sum_{\mu\in K_1} \mathrm{Re}\,(\beta_\mu m(\rho y)m(\rho\tau^{-1}b_\nu))$$
$$\cdot\sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu) - \sum_{\nu\in K_1} \mathrm{Im}\,\frac{|K_0|\beta_\nu}{Lm(\tau(x+y))}$$
$$\cdot\sum_{\rho\in K_1}\sum_{\mu\in K_1} \mathrm{Im}\,(\beta_\mu m(\rho y)m(\rho\tau^{-1}b_\nu))\sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu)$$

$$
\begin{aligned}
&= |K_0| \sum_{\rho\in K_1} \sum_{\mu\in K_1} \mathrm{Re}\ (\frac{\beta_\mu m(\rho y)}{Lm(\tau(x+y))} \sum_{\nu\in K_1} \beta_\nu m(\rho\tau^{-1}b_\nu)) \sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu)\\
&= |K_0|\cdot|K_1| \sum_{\mu\in K_1} \mathrm{Re}\ (\frac{\beta_\mu m(\tau y)}{Lm(\tau(x+y))}) \sum_{\lambda\in K_0} f(x+\lambda\tau^{-1}b_\mu)\\
&= |K_0|\cdot|K_1| \sum_{\mu\in K_1} \mathrm{Re}\ \frac{\beta_\mu}{Lm(\tau x)} \sum_{\lambda\in K_0} f(x+\lambda\tau^{-1}b_\mu)\\
&= |K_0|\cdot|K_1| g_\tau(x), \quad x\in G.
\end{aligned}
$$

and

$$
\begin{aligned}
&|K_1| \sum_{\lambda\in K_0} h_\tau(x+\lambda y)\\
&\quad = |K_1| \sum_{\lambda\in K_0} \sum_{\nu\in K_1} \mathrm{Im}\ \frac{\beta_\nu}{Lm(\tau(x+\lambda y))} \sum_{\sigma\in K_0} f(x+\lambda y+\sigma\tau^{-1}b_\nu)\\
&\quad = \sum_{\nu\in K_1} \mathrm{Im}\ \frac{\beta_\nu}{Lm(\tau(x+y))} \sum_{\sigma\in K_0} |K_1| \sum_{\lambda\in K_0} f(x+\lambda(y+\sigma\tau^{-1}b_\nu))\\
&\quad = \sum_{\nu\in K_1} \mathrm{Im}\ \frac{\beta_\nu}{Lm(\tau(x+y))} \sum_{\sigma\in K_0} \sum_{\rho\in K_1} \sum_{\mu\in K_1} \mathrm{Re}\ (\beta_\mu m(\rho(y+\sigma\tau^{-1}b_\nu)))\\
&\qquad \cdot \sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu)\\
&\quad = \sum_{\nu\in K_1} \mathrm{Im}\ \frac{|K_0|\beta_\nu}{Lm(\tau(x+y))} \sum_{\rho\in K_1} \sum_{\mu\in K_1} \mathrm{Re}\ (\beta_\mu m(\rho y) m(\rho\tau^{-1}b_\nu))\\
&\qquad \cdot \sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu) + \sum_{\nu\in K_1} \mathrm{Re}\ \frac{|K_0|\beta_\nu}{Lm(\tau(x+y))}\\
&\qquad \cdot \sum_{\rho\in K_1} \sum_{\mu\in K_1} \mathrm{Im}\ (\beta_\mu m(\rho y) m(\rho\tau^{-1}b_\nu)) \sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu)\\
&\quad = |K_0| \sum_{\rho\in K_1} \sum_{\mu\in K_1} \mathrm{Im}\ (\frac{\beta_\mu m(\rho y)}{Lm(\tau(x+y))} \sum_{\nu\in K_1} \beta_\nu m(\rho\tau^{-1}b_\nu)) \sum_{\lambda\in K_0} f(x+\lambda\rho^{-1}b_\mu)\\
&\quad = |K_0|\cdot|K_1| \sum_{\mu\in K_1} \mathrm{Im}\ (\frac{\beta_\mu m(\tau y)}{Lm(\tau(x+y))}) \sum_{\lambda\in K_0} f(x+\lambda\tau^{-1}b_\mu)\\
&\quad = |K_0|\cdot|K_1| \sum_{\mu\in K_1} \mathrm{Im}\ \frac{\beta_\mu}{Lm(\tau x)} \sum_{\lambda\in K_0} f(x+\lambda\tau^{-1}b_\mu)\\
&\quad = |K_0|\cdot|K_1| h_\tau(x), \quad x\in G.
\end{aligned}
$$

In view of Theorem 2, for each $\lambda \in K_1$ there exist $A_0^\lambda, B_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda, B_k^\lambda\colon S^k \to X, k \in \{1, \dots |K_0|-1\}$ such that

$$g_\lambda(x) = A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x), \quad x \in G,$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y \in G,\ 1 \le i \le k \le |K_0|-1,$$

$$h_\lambda(x) = B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x,\dots,x), \quad x \in G,$$

$$\sum_{\mu\in K_0} B_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y \in G,\ 1 \le i \le k \le |K_0|-1.$$

We observe that

$$\begin{aligned}
&\sum_{\tau\in K_1} \mathrm{Re}\ m(\tau x) g_\tau(x) - \sum_{\tau\in K_1} \mathrm{Im}\ m(\tau x) h_\tau(x)\\
&\quad= \sum_{\tau\in K_1}\Bigg[\mathrm{Re}\ m(\tau x) \sum_{\nu\in K_1} \mathrm{Re}\ \frac{\beta_\nu}{Lm(\tau x)} \sum_{\sigma\in K_0} f(x+\sigma\tau^{-1}b_\nu)\\
&\qquad -\mathrm{Im}\ m(\tau x) \sum_{\nu\in K_1} \mathrm{Im}\ \frac{\beta_\nu}{Lm(\tau x)} \sum_{\sigma\in K_0} f(x+\sigma\tau^{-1}b_\nu)\Bigg]\\
&\quad= \sum_{\tau\in K_1}\sum_{\nu\in K_1} \mathrm{Re}\ \left(m(\tau x)\frac{\beta_\nu}{Lm(\tau x)}\right) \sum_{\sigma\in K_0} f(x+\sigma\tau^{-1}b_\nu)\\
&\quad= \sum_{\tau\in K_1}\sum_{\sigma\in K_0}\sum_{\nu\in K_1} \mathrm{Re}\ \left(\frac{\beta_\nu}{L}\right) f(x+\sigma\tau^{-1}b_\nu)\\
&\quad= \sum_{\lambda\in K}\sum_{\nu\in K_1} \mathrm{Re}\ \left(\frac{\beta_\nu}{L}\right) f(x+\lambda b_\nu)\\
&\quad= \sum_{\nu\in K_1} \mathrm{Re}\ \beta_\nu\varphi(b_\nu) f(x) = m(0)f(x) = f(x), \quad x \in G,
\end{aligned}$$

which ends the proof. □

Corollary 1. *Functions $f, g\colon G \to X, \varphi\colon G \to \mathbb{K}, f \neq 0$, satisfy the equality*

$$\sum_{\lambda\in K} f(x+\lambda y) = L\varphi(y)g(x), \quad x,y \in G, \tag{21}$$

if and only if there exists a homomorphism $m\colon G \to \mathbb{C}^*$ such that

$$\varphi(x) = \varphi(0)\frac{1}{L}\sum_{\lambda\in K} m(\lambda x), \quad x \in G,$$

and (i) if X is real, then there exist $A_0^\lambda, B_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda, B_k^\lambda \colon G^k \to X, k \in \{1, \ldots |K_0|-1\}, \lambda \in K_1$ such that

$$\begin{aligned} f(x) &= \varphi(0) \sum_{\lambda \in K_1} \left(Re\ m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \ldots, x) \right] \right. \\ &\qquad \left. - Im\ m(\lambda x) \left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \ldots, x) \right] \right), \quad x \in G, \\ g(x) &= \sum_{\lambda \in K_1} \left(Re\ m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \ldots, x) \right] \right. \\ &\qquad \left. - Im\ m(\lambda x) [B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \ldots, x)] \right), \quad x \in G, \end{aligned}$$

$$\sum_{\mu \in K_0} A_k^\lambda\ (x, \ldots, x, \underbrace{\mu y, \ldots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \le i \le k \le |K_0| - 1,$$

$$\sum_{\mu \in K_0} B_k^\lambda\ (x, \ldots, x, \underbrace{\mu y, \ldots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \le i \le k \le |K_0| - 1,$$

(ii) if X is complex, then there exist $A_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda \colon G^k \to X, k \in \{1, \ldots |K_0|-1\}, \lambda \in K_1$ such that

$$f(x) = \varphi(0) \sum_{\lambda \in K_1} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \ldots, x) \right], \quad x \in G,$$

$$g(x) = \sum_{\lambda \in K_1} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \ldots, x) \right], \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \ldots, x, \underbrace{\mu y, \ldots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \le i \le k \le |K_0| - 1,$$

where $K_0 := \{\lambda \in K :\ m \circ \lambda = m\}$, $K_1 \subset K$ is a minimal set such that $K = K_0 \circ K_1$.

Proof. Putting $y = 0$ in (21) we have

$$Lf(x) = L\varphi(0)g(x), \quad x \in G.$$

Since $f \neq 0$, $g \neq 0$, $\varphi(0) \neq 0$ and

$$\varphi(0) \sum_{\lambda \in K} g(x + \lambda y) = L\varphi(y)g(x), \quad x, y \in G,$$

hence for $\varphi_0 := \frac{\varphi}{\varphi(0)}$ we have

$$\sum_{\lambda \in K} g(x+\lambda y) = L\varphi_0(y)g(x), \quad x, y \in G.$$

In view of Theorems 4 and 5 accordingly we obtain (ii) and (i) of the theorem. □

Theorem 6. *Functions* $f\colon G \to X, \varphi\colon G \to \mathbb{K}, f \neq 0, \varphi \neq const$, *satisfy the equality*

$$\sum_{\lambda \in K} f(x+\lambda y) = L\varphi(y)f(x) + \sum_{\lambda \in K} f(\lambda y), \quad x, y \in G, \tag{22}$$

if and only if there exists a homomorphism $m\colon G \to \mathbb{C}^*$ *such that*

$$\varphi(x) = \frac{1}{L}\sum_{\lambda \in K} m(\lambda x), \quad x \in G,$$

and
(i) if X *is real, then there exist* $A_0^\lambda, B_0^\lambda \in X$, *k-additive, symmetric mappings* $A_k^\lambda, B_k^\lambda\colon G^k \to X, k \in \{1, \dots |K_0|-1\}, \lambda \in K_1$ *such that*

$$\begin{aligned} f(x) = \sum_{\lambda \in K_1} \Bigg(Re\ m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] \\ - Im\ m(\lambda x) \left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x, \dots, x) \right] \Bigg) - \sum_{\lambda \in K_1} A_0^\lambda, \quad x \in G, \end{aligned}$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \leq i \leq k \leq |K_0|-1,$$

$$\sum_{\mu \in K_0} B_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \leq i \leq k \leq |K_0|-1,$$

(ii) if X *is complex, then there exist* $A_0^\lambda \in X$, *k-additive, symmetric mappings* $A_k^\lambda\colon G^k \to X, k \in \{1, \dots |K_0|-1\}, \lambda \in K_1$ *such that*

$$f(x) = \sum_{\lambda \in K_1} m(\lambda x) \left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x, \dots, x) \right] - \sum_{\lambda \in K_1} A_0^\lambda, \quad x \in G,$$

$$\sum_{\mu \in K_0} A_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \leq i \leq k \leq |K_0|-1,$$

where $K_0 := \{\lambda \in K :\ m \circ \lambda = m\}$, $K_1 \subset K$ *is a minimal set such that* $K = K_0 \circ K_1$.

Moreover

$$\sum_{\lambda\in K} f(\lambda x) = L(\varphi(x)-1)\sum_{\lambda\in K_1} A_0^\lambda, \quad x \in G. \tag{23}$$

Proof. Putting $x = 0$ in (22) we have

$$\sum_{\lambda\in K} f(\lambda y) = L\varphi(y)f(0) + \sum_{\lambda\in K} f(\lambda y), \quad y \in G.$$

Since $\varphi \neq const$, $f(0) = 0$. Putting $y = 0$ in (22) we get

$$Lf(x) = L\varphi(0)f(x) + Lf(0), \quad x \in G,$$

hence we obtain $\varphi(0) = 1$. We observe that

$$\begin{aligned} L\varphi(x)\sum_{\lambda\in K} f(\lambda y) + L\sum_{\lambda\in K} f(\lambda x) &= \sum_{\mu\in K}\sum_{\lambda\in K} f(\lambda y+\mu x) = \sum_{\mu\in K}\sum_{\lambda\in K} f(\mu x+\lambda y) \\ &= L\varphi(y)\sum_{\mu\in K} f(\mu x) + L\sum_{\lambda\in K} f(\lambda y), \quad x, y \in G. \end{aligned}$$

Hence we have

$$L(\varphi(y)-1)\sum_{\mu\in K} f(\mu x) = L(\varphi(x)-1)\sum_{\lambda\in K} f(\lambda y), \quad x, y \in G,$$

which means that

$$\sum_{\lambda\in K} f(\lambda y) = L(\varphi(y)-1)A_0, \quad y \in G, \tag{24}$$

for some $A_0 \in X$. Therefore we obtain that $f \neq const$. Putting the above equality in (22) we get

$$\sum_{\lambda\in K} f(x+\lambda y) = L\varphi(y)f(x) + L(\varphi(y)-1)A_0, \quad x, y \in G,$$

hence, for the function $g\colon G \to X$ given by the formula $g := f + A_0$, we obtain

$$\sum_{\lambda\in K} g(x+\lambda y) = L\varphi(y)g(x), \quad x, y \in G.$$

In view of Theorems 4 and 5 there exists a homomorphism $m\colon G \to \mathbb{C}^*$ such that

$$\varphi(x) = \frac{1}{L}\sum_{\lambda\in K} m(\lambda x), \quad x \in G,$$

and
(i) if X is real, then there exist $A_0^\lambda, B_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda, B_k^\lambda\colon G^k \to X, k \in \{1, \dots |K_0| - 1\}, \lambda \in K_1$ such that

$$g(x) = \sum_{\lambda\in K_1}\left(\mathrm{Re}\ m(\lambda x)\left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x)\right]\right.$$
$$\left.-\mathrm{Im}\ m(\lambda x)\left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x,\dots,x)\right]\right), \quad x\in G,$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

$$\sum_{\mu\in K_0} B_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

(ii) if X is complex, then there exist $A_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda\colon G^k \to X, k\in\{1,\dots|K_0|-1\}, \lambda\in K_1$ such that

$$g(x) = \sum_{\lambda\in K_1} m(\lambda x)\left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x)\right], \quad x\in G,$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

Since $f(0)=0$ and $g = f + A_0$, we have

$$A_0 = g(0) = \sum_{\lambda\in K_1} A_0^\lambda,$$

which together with (24) gives equality (23) and the form of the function f. □

Corollary 2. *Functions* $f\colon G\to X, \varphi\colon G\to\mathbb{K}, f\ne 0, \varphi\ne const$, *satisfy the equation*

$$\sum_{\lambda\in K} f(x+\lambda y) = L\varphi(y)f(x) + Lf(y), \quad x,y\in G, \tag{25}$$

if and only if there exist a homomorphism $m\colon G\to\mathbb{C}^*$, *and* $A\in X$ *such that*

$$\varphi(x) = \frac{1}{L}\sum_{\lambda\in K} m(\lambda x), \quad x\in G,$$
$$f(x) = (\varphi(x)-1)A, \quad x\in G.$$

Proof. It is easy to check that if functions f and φ satisfy the above conditions, then they satisfy Eq. (25).

Assume that f and φ satisfy Eq. (25). Putting $x=0$ in (25) we have

$$\sum_{\lambda\in K} f(\lambda y) = L\varphi(y)f(0) + Lf(y), \quad y\in G.$$

Putting $y=0$ in (25) we get

$$Lf(x)=L\varphi(0)f(x)+Lf(0),\quad x\in G.$$

If $\varphi(0)=0$, then $f=f(0)$ and from the above equalities $f=0$ which gives a contradiction. Hence, since $\varphi\neq const$, $f(0)=0$, $\varphi(0)=1$ and $\sum_{\lambda\in K}f(\lambda y)=Lf(y)$. In particular f and φ satisfy Eq. (22). In view of Theorem 6 we have

$$Lf(x)=\sum_{\mu\in K}f(\mu x)=L(\varphi(x)-1)A,\quad x,y\in G,$$

for some $A\in X$. □

Theorem 7. *Functions* $f,g,h\colon G\to X,\varphi\colon G\to\mathbb{K},f\neq 0,\varphi\neq const,\varphi(0)\neq 0,$ *satisfy the equation*

$$\sum_{\lambda\in K}f(x+\lambda y)=L\varphi(y)g(x)+Lh(y),\quad x,y\in G,\tag{26}$$

if and only if there exists a homomorphism $m\colon G\to\mathbb{C}^*,A,B\in X$ *such that*

$$\varphi(x)=\varphi(0)\frac{1}{L}\sum_{\lambda\in K}m(\lambda x),\quad x\in G,$$

and
(i) if X *is real, then there exist* $A_0^\lambda,B_0^\lambda\in X$, *k-additive, symmetric mappings* $A_k^\lambda,B_k^\lambda\colon G^k\to X,k\in\{1,\dots|K_0|-1\},\lambda\in K_1$ *such that*

$$\begin{aligned}f(x)=\varphi(0)\sum_{\lambda\in K_1}&\left(Re\ m(\lambda x)\left[A_0^\lambda+\sum_{i=1}^{|K_0|-1}A_i^\lambda(x,\dots,x)\right]\right.\\&\left.-Im\ m(\lambda x)\left[B_0^\lambda+\sum_{i=1}^{|K_0|-1}B_i^\lambda(x,\dots,x)\right]\right)+A-\varphi(0)\sum_{\lambda\in K_1}A_0^\lambda,\quad x\in G,\\g(x)=\sum_{\lambda\in K_1}&\left(Re\ m(\lambda x)\left[A_0^\lambda+\sum_{i=1}^{|K_0|-1}A_i^\lambda(x,\dots,x)\right]\right.\\&\left.-Im\ m(\lambda x)\left[B_0^\lambda+\sum_{i=1}^{|K_0|-1}B_i^\lambda(x,\dots,x)\right]\right)+B-\sum_{\lambda\in K_1}A_0^\lambda,\quad x\in G,\\h(x)=\varphi(x)&\left(\sum_{\lambda\in K_1}A_0^\lambda-B\right)+\left(A-\varphi(0)\sum_{\lambda\in K_1}A_0^\lambda\right),\quad x\in G,\end{aligned}$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\ldots,x,\underbrace{\mu y,\ldots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

$$\sum_{\mu\in K_0} B_k^\lambda(x,\ldots,x,\underbrace{\mu y,\ldots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

(ii) if X is complex, then there exist $A_0^\lambda \in X$, k-additive, symmetric mappings $A_k^\lambda\colon G^k \to X, k\in\{1,\ldots|K_0|-1\}, \lambda\in K_1$ such that

$$f(x)=\varphi(0)\left(\sum_{\lambda\in K_1} m(\lambda x)\left[A_0^\lambda+\sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\ldots,x)\right]-\sum_{\lambda\in K_1} A_0^\lambda\right)+A, \quad x\in G,$$

$$g(x) = \sum_{\lambda\in K_1} m(\lambda x)\left[A_0^\lambda+\sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\ldots,x)\right]+B-\sum_{\lambda\in K_1} A_0^\lambda, \quad x\in G,$$

$$h(x) = \varphi(x)\left(\sum_{\lambda\in K_1} A_0^\lambda - B\right)+\left(A-\varphi(0)\sum_{\lambda\in K_1} A_0^\lambda\right), \quad x\in G,$$

$$\sum_{\mu\in K_0} A_k^\lambda(x,\ldots,x,\underbrace{\mu y,\ldots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

where $K_0 := \{\lambda\in K:\ m\circ\lambda = m\}$, $K_1 \subset K$ is a minimal set such that $K = K_0\circ K_1$.

Proof. Putting $x=0$ in (26) we have

$$\sum_{\lambda\in K} f(\lambda y) = L\varphi(y)g(0)+Lh(y), \quad y\in G.$$

Putting $y=0$ in (26) we get

$$Lf(x) = L\varphi(0)g(x)+Lh(0), \quad x\in G.$$

Hence we have

$$\begin{aligned}\varphi(0)\sum_{\lambda\in K} g(x+\lambda y) &= \sum_{\lambda\in K} f(x+\lambda y) - Lh(0)\\ &= L\varphi(y)g(x)+Lh(y)-Lh(0)\\ &= L\varphi(y)g(x)+\sum_{\lambda\in K} f(\lambda y) - L\varphi(y)g(0)-Lh(0)\\ &= L\varphi(y)(g(x)-g(0))+\varphi(0)\sum_{\lambda\in K} g(\lambda y), \quad x,y\in G,\end{aligned}$$

which gives us that functions $g_0 = g-g(0)$, $\varphi_0 = \frac{\varphi}{\varphi(0)}$ satisfy the equation

$$\sum_{\lambda\in K} g_0(x+\lambda y) = L\varphi_0(y)g_0(x)+\sum_{\lambda\in K} g_0(\lambda y), \quad x,y\in G.$$

In view of Theorem 6 there exists a homomorphism $m\colon G\to\mathbb{C}^*$ such that

$$\varphi_0(x)=\frac{1}{L}\sum_{\lambda\in K} m(\lambda x),\quad x\in G,$$

and
(i) if X is real, then there exist $A_0^\lambda, B_0^\lambda\in X$, k-additive, symmetric mappings $A_k^\lambda, B_k^\lambda\colon G^k\to X, k\in\{1,\dots|K_0|-1\},\lambda\in K_1$ such that

$$\begin{aligned}g_0(x)=\sum_{\lambda\in K_1}\Bigg(&\mathrm{Re}\, m(\lambda x)\left[A_0^\lambda+\sum_{i=1}^{|K_0|-1}A_i^\lambda(x,\dots,x)\right]\\ &-\mathrm{Im}\, m(\lambda x)\left[B_0^\lambda+\sum_{i=1}^{|K_0|-1}B_i^\lambda(x,\dots,x)\right]\Bigg)-\sum_{\lambda\in K_1}A_0^\lambda,\quad x\in G,\end{aligned}$$

$$\sum_{\mu\in K_0}A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i})=0,\quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le|K_0|-1,$$

$$\sum_{\mu\in K_0}B_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i})=0,\quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le|K_0|-1,$$

(ii) if X is complex, then there exist $A_0^\lambda\in X$, k-additive, symmetric mappings $A_k^\lambda\colon G^k\to X, k\in\{1,\dots|K_0|-1\},\lambda\in K_1$ such that

$$g_0(x)=\sum_{\lambda\in K_1}m(\lambda x)[A_0^\lambda+\sum_{i=1}^{|K_0|-1}A_i^\lambda(x,\dots,x)]-\sum_{\lambda\in K_1}A_0^\lambda,\quad x\in G,$$

$$\sum_{\mu\in K_0}A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i})=0,\quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k\le|K_0|-1,$$

Moreover

$$\sum_{\lambda\in K}g_0(\lambda x)=L(\varphi_0(x)-1)\sum_{\lambda\in K_1}A_0^\lambda,\quad x\in G.$$

Hence, putting $B:=g(0)$, we obtain the form of φ and g. Since

$$Lf(x)=L\varphi(0)g(x)+Lh(0)=L\varphi(0)g_0(x)+Lf(0),\quad x\in G,$$

and

$$\begin{aligned}Lh(x)&=\sum_{\lambda\in K}f(\lambda x)-L\varphi(x)g(0)=\varphi(0)\sum_{\lambda\in K}g(\lambda x)+Lh(0)-L\varphi(x)g(0)\\ &=\varphi(0)\sum_{\lambda\in K}g_0(\lambda x)+Lf(0)-L\varphi(x)g(0),\quad x\in G,\end{aligned}$$

from the form of g_0 we obtain the form of f and h in the real and the complex case of the space X. $\square$

References

[1] Aczél, J., Dhombres, J.: Functional Equations in Several Variables. Cambridge University Press, Cambridge (1989)
[2] Badora, R.: On a generalized Wilson functional equation. Georgian Math. J. **12**(4), 595–606 (2005)
[3] Chojnacki, W.: Group representations of bounded cosine functions. Technical report of the University of Adelaide, TR9409 (1994)
[4] Czerwik, S.: On the stability of the quadratic mappings in normed spaces. Abh. Math. Semin. Univ. Hamb. **62**, 59–64 (1992)
[5] Förg-Rob, W., Schwaiger, J.: A generalization of the cosine equation to n summands. Grazer Math. Ber. **316**, 219–226 (1992)
[6] Gajda, Z.: A remark on the talk of W. Förg-Rob. Grazer Math. Ber **316**, 234–237 (1992)
[7] Kannappan, P.L.: The functional equation $f(xy) + f(xy^{-1}) = 2f(x)f(y)$ for groups. Proc. Am. Math. Soc **19**, 69–74 (1968)
[8] Łukasik, R.: Some generalization of Cauchy's and the quadratic functional equations. Aequationes Math. **83**, 75–86 (2012)
[9] Shinya, H.: Spherical matrix functions and Banach representability for locally compact motion groups. Jpn. J. Math. **28**, 163–201 (2002)
[10] Sinopoulos, P.: Functional equations on semigroups. Aequationes Math. **59**, 255–261 (2000)
[11] Stetkær, H.: On a signed cosine equation of N summands. Aequationes Math. **51**(3), 294–302 (1996)
[12] Stetkær, H.: Wilsons functional equation on C. Aequationes Math. **53**(1-2), 91–107 (1997)
[13] Stetkær, H.: Functional equation on abelian groups with involution, II. Aequationes Math. **55**, 227–240 (1998)
[14] Stetkær, H.: Functional equations involving means of functions on the complex plane. Aequationes Math **56**, 47–62 (1998)

Radosław Łukasik
Institute of Mathematics
University of Silesia
ul. Bankowa 14
40-007 Katowice
Poland
e-mail: rlukasik@math.us.edu.pl

Received: September 17, 2012